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On Characterizations of the Gamma Function

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1. Introduction It is well known that the gamma function $\Gamma(x) > 0$ on $(0, \infty)$ satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ and the initial condition $\Gamma(1) = 1$. However, these two properties do not characterize the gamma function. Rather surprisingly, the additional assumption of the convexity of $\log \Gamma(x)$ is sufficient for a characterization, a fact discovered by Bohr and Mollerup [**1**]. For a proof, see Artin's book [**4**, **5**] or Rudin's book [**6**], or the last section of this paper. Note that the initial condition in the characterization is not essential, for if f is a positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$ then $g(x) = f(1)^{-1}f(x)$ is a positive function that satisfies the same functional equation and $g(1) = 1$.

A second characterization formulated and proved by Laugwitz and Rodewald [**2**] says that the convexity of $\log \Gamma(x)$ can be replaced by the property, call it property (L), that the function $L(x) = \log \Gamma(x+1)$ satisfies

$$L(n+x) = L(n) + x\log(n+1) + r_n(x), \quad \text{where } r_n(x) \to 0 \text{ as } n \to \infty. \qquad \text{(L)}$$

However, they did not show how this property is related to the convexity of $\log \Gamma(x)$. The original idea of the second characterization goes back to Euler [**3**].

In the present paper we give a third characterization of the gamma function and then show how these three characterizations are related.

2. A third characterization In property (L), the use of logarithms is not essential and without logarithms the expression on the right-hand side becomes a product instead of a sum. We might therefore expect that a modified property (L) will give us a characterization that is closer to the product expression of the gamma function. With this in mind, we modify property (L) as follows: The gamma function satisfies the following property

$$\Gamma(x+n) = \Gamma(n)n^x t_n(x), \quad \text{where } t_n(x) \to 1 \text{ as } n \to \infty.$$

THEOREM 1. *There exists a unique function $f(x) > 0$ on $(0,\infty)$ that satisfies the following three properties:*

(a) $f(1) = 1$;
(b) $f(x+1) = xf(x)$;
(c) $f(x+n) = f(n)n^x t_n(x)$, *where* $t_n(x) \to 1$ *as* $n \to \infty$.

DEFINITION. *For each positive integer n, we define the function* Γ_n *on* $(0,\infty)$ *by*

$$\Gamma_n(x) = \frac{n^x n!}{x(x+1)\cdots(x+n)}, \quad x > 0.$$

LEMMA. *The sequence* $(\Gamma_n(x))$ *of functions on* $(0,\infty)$ *converges for any* $x > 0$.

Proof. Taking logarithms, we have

$$\begin{aligned}\log \Gamma_n(x) &= x \log n + \sum_{k=1}^{n} \log k - \log x - \sum_{k=1}^{n} \log(x+k)\\ &= x \log n - \log x - \sum_{k=1}^{n} \log\left(1 + \frac{x}{k}\right)\\ &= -\log x - x\left[\sum_{k=1}^{n} \frac{1}{k} - \log n\right] + \sum_{k=1}^{n}\left[\frac{x}{k} - \log\left(1+\frac{x}{k}\right)\right]\\ &= -\log x - x\gamma_n + c_n(x),\end{aligned}$$

where $\gamma_n = \sum_{k=1}^{n}\frac{1}{k} - \log n$, and $c_n(x) = \sum_{k=1}^{n}[\frac{x}{k} - \log(1+\frac{x}{k})]$. It is well known that (γ_n) converges to Euler's constant $\gamma \approx 0.577\ldots$. Also, the sequence $(c_n(x))$ converges, since for $k > x > 0$,

$$0 < \frac{x}{k} - \log\left(1 + \frac{x}{k}\right) = \frac{x}{k} - \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}\left(\frac{x}{k}\right)^i \le \frac{x^2}{2k^2}.$$

Thus, the sequence $(\log \Gamma_n(x))$ converges and hence so does the sequence $(\Gamma_n(x))$ for $x > 0$. This completes the proof of the lemma.

Remark. In fact, the limit function of the above sequence is the product expression of the gamma function (see [4, 5]). Therefore we have

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}, \quad \text{for } x > 0. \tag{1}$$

Proof of Theorem 1. First we prove that $\Gamma(x)$ in (1) satisfies (a)–(c).
(a) $\Gamma(1) = 1$, since

$$\Gamma(1) = \lim_{n\to\infty} \frac{n^1 n!}{1(1+1)\cdots(1+n)} = \lim_{n\to\infty} \frac{n}{1+n} = 1.$$

(b) Γ satisfies the functional equation, since

$$\begin{aligned}\Gamma(x+1) &= \lim_{n\to\infty} \frac{n^{x+1} n!}{(x+1)(x+2)\cdots(x+1+n)}\\ &= \lim_{n\to\infty} \frac{nx}{x+1+n} \cdot \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}\\ &= x\Gamma(x).\end{aligned}$$

As a consequence of these two properties, we get $\Gamma(n)=(n-1)!$.

(c) Let $s_n(x)=\Gamma(x)/\Gamma_n(x)$. Then $\Gamma(x)=\Gamma_n(x)s_n(x)$, and $\lim_{n\to\infty} s_n(x)=1$.

For natural n and real $x>0$, we apply (b) n times to get

$$\begin{aligned}\Gamma(x+n) &= [(x+n-1)\cdots(x+1)x]\cdot\Gamma(x)\\ &= \frac{(x+n)\cdots(x+1)x}{x+n}\cdot\frac{n^x n!}{x(x+1)\cdots(x+n)}\cdot s_n(x)\\ &= n^x\Gamma(n)t_n(x),\end{aligned}$$

where $t_n(x)=(n/(x+n))\cdot s_n(x)$. Thus, $\Gamma(x+n)=n^x\Gamma(n)t_n(x)$ and $t_n(x)\to 1$ as $n\to\infty$.

To show uniqueness, we assume $f(x)$ is a function that satisfies (a)–(c). From properties (a) and (b), we have

$$f(n)=(n-1)!, \tag{2}$$

$$f(x+n)=(x+n-1)(x+n-2)\cdots(x+1)xf(x). \tag{3}$$

Combining (3), property (c), and (2) together, we have

$$f(x)=\frac{x^n(n-1)!}{x(x+1)\cdots(x+n-1)}\cdot t_n(x)=\Gamma_n(x)\cdot s_n(x),$$

where $s_n(x)=((x+n)/n)t_n(x)\to 1$ as $n\to\infty$. Therefore $f(x)=\Gamma(x)$ and hence f is uniquely determined. This completes the proof of the third characterization of the gamma function.

3. How are these characterizations related? To simplify our discussion, we adopt the following terminology.

DEFINITION. *By a PG function (pre-gamma function), we mean a positive function f on $(0,\infty)$ that satisfies the functional equation $f(x+1)=xf(x)$.*

Remark. For a PG function f, we may assume $f(1)=1$, since if g is a PG function then $f(x)=g(1)^{-1}g(x)$ is also a PG function such that $f(1)=1$. Now we can rephrase what we have so far on characterizations of the gamma function.

CHARACTERIZATIONS. *If f is a PG function such that*

(C) $$\log f \text{ is convex on } (0,\infty),$$

or

(L) $$L(n+x)=L(n)+x\log(n+1)+r_n(x),$$

where $L(x)=\log f(x+1)$ and $r_n(x)\to 0$ as $n\to\infty$, or

(P) $$f(n+x)=f(n)n^x t_n(x),$$

where $t_n(x)\to 1$ as $n\to\infty$, then $f(x)=c\Gamma(x)$, for some constant c.

Remarks. (a) It is easy to see that the constant c in each characterization is simply $f(1)$. In other words, any PG function f with $f(1)=1$ that satisfies either (C) or (L) or (P) must be the gamma function.

(b) In the previous section we showed that (P) characterizes the gamma function. Therefore it suffices to show that properties (C), (L), and (P) are equivalent to one another for a PG function. To do this, we need three basic facts about convex functions (see [4, 5]).

(1) *If f is convex on (a, b) and if $x < y$, $x, y \in (a, b)$, then*

$$\frac{f(x)-f(c)}{x-c} \le \frac{f(y)-f(c)}{y-c}$$

for any $c \in (a, b)$.

(2) *The limit function of a convergent sequence of convex functions is convex.*

(3) *Let g be a twice-differentiable function on (a, b). Then g is convex on (a, b) if, and only if, $g''(x) > 0$ for all $x \in (a, b)$.*

THEOREM 2. *For a PG function f, the properties* (C), (L), *and* (P) *are equivalent.*

Proof. (a) (P) $\Leftrightarrow$ (L). We have

$$\begin{aligned}
&(\mathrm{P})\\
&\Leftrightarrow f(x+(n+1)) = f(n+1)(n+1)^x t_{n+1}(x),\ t_{n+1}(x) \to 1\\
&\Leftrightarrow \log f((x+n)+1) = \log f(n+1) + x\log(n+1) + \log t_{n+1}(x),\ t_{n+1}(x) \to 1\\
&\Leftrightarrow L(x+n) = L(n) + x\log(n+1) + r_n(x),\ r_n(x) \to 0\\
&\Leftrightarrow (\mathrm{L}).
\end{aligned}$$

(b) (C) $\Rightarrow$ (P). Let $m < x \le m+1$, where $m = 0, 1, 2\dots$. For any natural n, $n+m-1 < n+m < n+x \le n+m+1$. The convexity of $\log f$ and (1) above give us (we write $L_m = \log f(n+m)$)

$$\begin{aligned}
&\frac{L_m - L_{m-1}}{n+m-(n+m-1)} \le \frac{\log f(n+x) - \log f(n+m)}{(n+x)-(n+m)} \le \frac{L_{m+1}-L_m}{(n+m+1)-(n+m)}\\
&\quad\Leftrightarrow (x-m)\log(n+m-1) \le \log\left(\frac{f(n+x)}{f(n+m)}\right) \le (x-m)\log(n+m)\\
&\quad\Leftrightarrow (n+m-1)^{x-m} \le \frac{f(n+x)}{(n+m-1)(n+m-2)\cdots nf(n)} \le (n+m)^{x-m}\\
&\quad\Leftrightarrow \left(1+\frac{m-1}{n}\right)^x T_{m-1} \le \frac{f(n+x)}{f(n)n^x} \le \left(1+\frac{m}{n}\right)^x T_m,
\end{aligned}$$

where

$$T_m = \left(1-\frac{1}{n+m}\right)\left(1-\frac{2}{n+m}\right)\cdots\left(1-\frac{m}{n+m}\right).$$

Therefore, we have

$$\lim_{n\to\infty} \frac{f(n+x)}{f(n)n^x} = 1,$$

by the squeezing theorem. If we let

$$t_n(x) = \frac{f(n+x)}{f(n)n^x},$$

then

$$f(n+x) = f(n)n^x t_n(x),$$

where $t_n(x) \to 1$ as $n \to \infty$. This proves that f satisfies (P).

(c) (P) $\Rightarrow$ (C). From the uniqueness part of the proof of Theorem 1 we have

$$f(x) = f(1) \lim_{n \to \infty} \Gamma_n(x).$$

By (2) above, it suffices to show that $\log \Gamma_n(x)$ is convex. Now

$$(\log \Gamma_n(x))'' = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \cdots + \frac{1}{(x+n)^2} > 0.$$

By (3) above, $\log \Gamma_n(x)$ is convex and so is $\log f$. This completes the proof.

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