3rd Exam for Calculus II 4181

Name : ________________ Student ID # : ________________ Score : __________

1. Compute the value of the limit or prove that the limit does not exist:

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^2}{x^4 + y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4} = 1$$

$$\nparallel$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^4-y^2}{x^4+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^2}{x^4+y^2}$ does not exist #

(b) $\lim\limits_{(x,y)\to(0,0)} x\cos\dfrac{1}{y}$

$$-1 \le \cos\frac{1}{y} \le 1 \Rightarrow -|x| \le x\cos\frac{1}{y} \le |x|$$

$$\because \lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$$

By Sandwich Thm $\lim\limits_{(x,y)\to(0,0)} x\cos\dfrac{1}{y} = 0$ #

2. Find all the second partial derivatives of the function $f(x,y) = x^y$.

$f_x = y x^{y-1}$ $\qquad$ $f_y = x^y \ln x$

$f_{xx} = y(y-1)x^{y-2}$

$f_{xy} = x^{y-1} + y \cdot x^{y-1} \ln x$

$f_{yx} = y x^{y-1} \ln x + x^y \cdot \frac{1}{x}$

$f_{yy} = x^y (\ln x)^2$

3. Suppose f is a differentiable function and $f(1,0) = 2$, $f_x(1,0) = 7$, $f_y(1,0) = 11$. Let $w = f(st, t^2 - s^2)$. Calculate the following partial derivatives:

(a) $\frac{\partial w}{\partial s}(1,1)$

Let $x = st$, $y = t^2 - s^2$ $\left(\begin{array}{l} s=1,\ t=1 \\ \Rightarrow x=1,\ y=0 \end{array} \right)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = f_x(x,y)\cdot t + f_y(x,y)(-2s)$$

$$\frac{\partial w}{\partial s}(1,1) = f_x(1,0)\cdot 1 + f_y(1,0)(-2) = 7 + 11(-2) = 7 - 22 = -15$$

$\begin{pmatrix} s=1 \\ t=1 \end{pmatrix}$

(b) $\frac{\partial w}{\partial t}(1,1)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = f_x(x,y)\cdot s + f_y(x,y)(2t)$$

$$\frac{\partial w}{\partial t}(1,1) = f_x(1,0)\cdot 1 + f_y(1,0)\cdot 2 = 7 + 11\cdot 2 = 7 + 22 = 29$$

4. Directional derivatives:

(a) Calculate the directional derivative of the function $f(x,y) = 2xy - 3y^2$ in the direction of $\vec{\mathbf{u}} = 4\vec{i} + 3\vec{j}$ at the point $(5,5)$.

$$\vec{u} = \frac{(4,3)}{\sqrt{4^2+3^2}} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$D_u f(5,5) = \nabla f(5,5)\cdot\vec{u}$$
$$= (10,-20)\cdot\left(\frac{4}{5}, \frac{3}{5}\right)$$
$$= 8 - 12 = -4$$

$$\nabla f = (f_x, f_y)$$
$$= (2y, 2x - 6y)$$
$$\nabla f(5,5) = (10,-20)$$

(b) Let $f(x,y) = x^2 - xy + y^2 - y$. Find the direction $\vec{\mathbf{u}}$ and the value $D_{\vec{\mathbf{u}}}f(1,-1)$ for which $D_{\vec{\mathbf{u}}}f(1,-1)$ is largest.

$$\nabla f(1,-1) = (f_x, f_y)\big|_{(1,-1)} = (2x - y, -x + 2y - 1)\big|_{(1,-1)} = (3,-4)$$

Max $D_u f(1,-1)$ occurs when $\vec{u} = \frac{\nabla f}{|\nabla f|}$

$$\therefore \vec{u} = \frac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = \frac{(3,-4)}{\sqrt{3^2+(-4)^2}} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

5. Find the equation for the tangent plane at the point $(0, 1, 2)$ on the level surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$.

Let $F(x,y,z) = \cos\pi x - x^2 y + e^{xz} + yz = 4$ be a level surface

$\Rightarrow$ T.P at $(0,1,2)$ is $\nabla F(0,1,2) \cdot (x-0, y-1, z-2) = 0$

$\nabla F = (F_x, F_y, F_z) = (-\pi\sin\pi x - 2xy + ze^{xz}, -x^2+z, xe^{xz}+y)$

$\Rightarrow \nabla F(0,1,2) = (2,2,1)$

$\therefore$ T.P is $(2,2,1)\cdot(x-0, y-1, z-2) = 0$

$\Rightarrow 2x + 2(y-1) + (z-2) = 0$

or $2x + 2y + z - 4 = 0$

6. The surface area S (measured in m^2) of a human body can be expressed as a function of its weight (measured in kg) and its height (measured in cm) by

$$S(x, y) = 0.007184x^{0.425}y^{0.725}.$$

If the percentage error for measuring weight is 1% and the percentage error for measuring height is 2%. Use the differntial to estimate the percentage error for calculating the surface area S according to the above formula.

Let $a = 0.007184$, $b = 0.425$, $c = 0.725$, $\frac{dx}{x} = \frac{1}{100}$, $\frac{dy}{y} = \frac{2}{100}$

$\Rightarrow S = ax^b y^c$

$dS = S_x dx + S_y dy$

$= S_x \cdot x \frac{dx}{x} + S_y \cdot y \frac{dy}{y}$

$= abx^{b-1}y^c \cdot x \cdot \frac{1}{100} + acx^b y^{c-1} \cdot y \cdot \frac{2}{100} = b \cdot S \cdot \frac{1}{100} + c \cdot S \cdot \frac{2}{100}$

$\frac{dS}{S} = b \cdot \frac{1}{100} + c \cdot \frac{2}{100} = 0.425 \cdot \frac{1}{100} + 0.725 \frac{2}{100} = 1.875\%$

7. Find all critical points of the function $f(x,y) = 4xy - x^4 - y^4$, and classify them as a local maximum, a local minimum or a saddle point.

$$\begin{cases} f_x = 4y - 4x^3 = 0 \\ f_y = 4x - 4y^3 = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases} \Rightarrow y = y^9 \Rightarrow y^9 - y = 0$$

$$\Rightarrow y(y^4+1)(y^2+1)(y-1)(y+1) = 0$$

$$\Rightarrow y = 0, 1, -1$$

c.p $(0,0)$ $(1,1)$ $(-1,-1)$

$f_{xx} = -12x^2$, $f_{xy} = 4$, $f_{yy} = -12y^2$

$D(x,y) = (-12x^2)(-12y^2) - 4^2 = 144x^2y^2 - 16$

$D(0,0) = -16 < 0 \Rightarrow (0,0,0)$ is a saddle point.

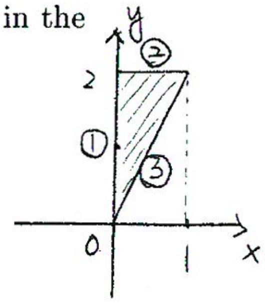
$D(1,1) = 144 - 16 > 0$ & $f_{xx} = -12 < 0$ $\therefore$ $(1,1,2)$ is a l. max

$D(-1,-1) = 144 - 16 > 0$ & $f_{xx} = -12 < 0$ $\therefore$ $(-1,-1,2)$ is a l. max

8. Find the absolute maximum and minimum of the function

$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

on the closed triangular plate bounded by the line $x = 0, y = 2, y = 2x$ in the first quadrant.



$$\begin{cases} f_x = 4x - 4 = 0 \\ f_y = 2y - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases} \qquad \text{c.p}(1,2) \quad f(1,2) = -5$$

On the boundary

① $x = 0$ $0 \le y \le 2$

$f(0,y) = y^2 - 4y + 1$, $f' = 2y - 4 = 0 \Rightarrow y = 2 \Rightarrow \begin{cases} (0,0) \\ (0,2) \end{cases}$ $f(0,0) = 1$ ← Max, $f(0,2) = -3$

② $y = 2$ $0 \le x \le 1$

$f(x,2) = 2x^2 - 4x - 3$ $f' = 4x - 4 = 0 \Rightarrow x = 1$ $\begin{cases} (1,2) \\ (0,2) \end{cases}$ $f(1,2) = -5$ ← min

③ $y = 2x$ $0 \le x \le 1$

$f(x,2x) = 2x^2 - 4x + 4x^2 - 8x + 1 \Rightarrow f' = 12x - 12 \Rightarrow x = 1$ $\begin{cases} (1,2) \times \\ (0,0) \checkmark \end{cases}$

$= 6x^2 - 12x + 1$