Key to 1st Midterm Exam for Calculus A2-EE

Part A **Multiple-Choice (20%)**

1. If $f'(x) = -f(x)$ and $f(1) = 1$, then $f(x) =$
   (A) $\frac{1}{2}e^{-2x+2}$ (B) e^{-x-1} (C) e^{1-x} (D) e^{-x} (E) $-e^{x}$

   Solution: Write the DE as $f'(x) + f(x) = 0$. It's linear with constant coefficients. Since its characteristic equation is $r + 1 = 0$, the general solution is $f(x) = Ce^{-x}$. But the initial condition $f(1) = 1$ gives us that $Ce^{-1} = f(1) = 1 \Rightarrow C = e$, and hence $f(x) = ee^{-x} = e^{1-x}$, which is (C).

2. If $y'' = 2y'$ and if $y = y' = e$ when $x = 0$, then when $x = 1, y =$
   (A) $\frac{e}{2}(e^2 + 1)$ (B) e (C) $\frac{e^3}{2}$ (D) $\frac{e}{2}$ (E) $\frac{e^3 - e}{2}$

   Solution: Write the DE as $y'' - 2y' = 0$. It's linear with constant coefficients. Since its characteristic equation is $r^2 - 2r = 0 \Rightarrow r = 0, 2$; the general solution is $y = C_1 + C_2e^{2x}$. Clearly, we have $y' = 2C_2e^{2x}$; therefore the initial conditions $y(0) = y'(0) = e$ give us that $C_1 + C_2 = y(0) = e$ and $2C_2 = y'(0) = e \Rightarrow C_2 = e/2$, and so $C_1 = e/2$. Put everything together, we have $y = \frac{e}{2} + \frac{e}{2}e^{2x} \Rightarrow y(1) = \frac{e}{2}(e^2 + 1)$, which is (A).

3. If f is the solution of $xf'(x) - f(x) = x$ such that $f(-1) = 1$, then $f(e^{-1}) =$
   (A) $-2e^{-1}$ (B) 0 (C) e^{-1} (D) $-e^{-1}$ (E) $2e^{-2}$

   Solution: Write the DE as $f'(x) - \frac{1}{x}f(x) = 1$. It's linear 1st order. An integrating factor is

   $$e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

   So we have $(\frac{1}{x}f(x))' = \frac{1}{x} \Rightarrow \frac{1}{x}f(x) = \ln|x| + C \Rightarrow f(x) = x\ln|x| + Cx$. But the initial condition $f(-1) = 1$ gives us that $-C = f(-1) = 1 \Rightarrow C = -1$, and hence $f(x) = x\ln|x| - x \Rightarrow f(e^{-1}) = e^{-1}\ln|e^{-1}| - e^{-1} = -2e^{-1}$, which is (A).

4. If $f''(x) - f'(x) - 2f(x) = 0, f'(0) = -2$, and $f(0) = 2$, then $f(1) =$
   (A) $e^2 + e^{-1}$ (B) 1 (C) 0 (D) e^2 (E) $2e^{-1}$

   Solution: The DE is linear with constant coefficients. Since its characteristic equation is $r^2 - r - 2 = 0 \Rightarrow r = -1, 2$; the general solution is $f(x) = C_1e^{-x} + C_2e^{2x}$. Clearly, we have $f'(x) = -C_1e^{-x} + 2C_2e^{2x}$; therefore the initial conditions $f(0) = 2, f'(0) = -2$ give us that

   $$C_1 + C_2 = f(0) = 2 \qquad \text{and} \qquad -C_1 + 2C_2 = f'(0) = -2.$$

   Add them up, we have $3C_2 = 0 \Rightarrow C_2 = 0$ and hence $C_1 = 2$. So we have $f(x) = 2e^{-x} \Rightarrow f(1) = 2e^{-1}$, which is (E).

Part B **Free-Response Questions (80%)**

1. Find the sum of the series: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

   Solution: Start from $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}, \quad |x| < 1.$

   Differentiating both sides with respect to x, we have $\displaystyle\sum_{n=0}^{\infty} nx^{n-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$;

   then multiplying both sides by x yields that $\displaystyle\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

   Set $x = \frac{1}{3}$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}$.

2. Solve the DE: $y' + 2y = 2e^x$

   Solution: The DE is linear and of 1st order. An integrating factor is $e^{\int 2\,dx} = e^{2x}$.
   Multiplying both sides by e^{2x} yields that $(e^{2x}y)' = 2e^{3x}$, and hence $e^{2x}y = \frac{2}{3}\,e^{3x} + C$.
   Therefore the general solution of the DE is $y = \frac{2}{3}\,e^x + Ce^{-2x}$.

3. Solve the initial-value problem: $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

   Solution: Write the DE as $y' - \frac{1}{x}y = x \sin x$.
   It's linear and of 1st order. An integrating factor is
   $$e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$
   So we have $(\frac{1}{x}y)' = \sin x \Rightarrow \frac{1}{x}y = -\cos x + C \Rightarrow y = -x\cos x + Cx$.
   But the initial condition $y(\pi) = 0$ gives us that $-\pi\cos\pi + C\pi = y(\pi) = 0 \Rightarrow C = -1$,
   and hence the solution to the initial-value problem is $y = -x\cos x - x$.

4. Solve the initial-value problem: $2y'' + 5y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -4$

   Solution: The DE is linear with constant coefficients. Since its characteristic equation is
   $$r^2 - 4r + 5 = 0 \Rightarrow r = -1, -\frac{3}{2}$$
   the general solution is $y = C_1e^{-x} + C_2e^{-\frac{3}{2}x}$.Clearly, we have $y' = -C_1e^{-x} - \frac{3}{2}\,C_2e^{-\frac{3}{2}x}$;
   therefore the initial conditions $y(0) = 3, \quad y'(0) = -4$ give us that
   $$C_1 + C_2 = y(0) = 3 \qquad \text{and} \qquad -C_1 - \frac{3}{2}C_2 = y'(0) = -4.$$
   Add them up, we have $-\frac{1}{2}C_2 = -1 \Rightarrow C_2 = 2$ and hence $C_1 = 1$.
   So the solution to the initial-value problem is $e^{-x} + 2e^{-\frac{3}{2}x}$.

5. Solve the DE using the method of undetermined coefficients: $y'' - 4y' + 5y = e^{-x}$

   Solution: The DE is linear with constant coefficients. Since its characteristic equation is $2r^2+5r+3 = 0 \Rightarrow r = 2 \pm \sqrt{-1}$; the complementary function is $y_c = C_1e^{2x}\cos x + C_2e^{2x}\sin x$. Clearly, a particular solution is of the form $y_p = Ae^{-x}$; and therefore $y_p' = -Ae^{-x} \quad y_p'' = Ae^{-x}$. $y_p'' - 4y_p' + 5y_p = e^{-x}$ gives us that
   $$Ae^{-x} - 4(-Ae^{-x}) + 5Ae^{-x} = e^{-x} \Rightarrow 10A = 1 \Rightarrow A = \frac{1}{10}\,.$$
   Conclude that the solution is $y = y_c + y_p = C_1e^{2x}\cos x + C_2e^{2x}\sin x + \dfrac{1}{10}\,e^{-x}$.

6. Let f and g be functions that are differentiable for all real number x and that have the following properties: (i) $f'(x) = f(x) - g(x)$ (ii) $g'(x) = g(x) - f(x)$ (iii) $f(0) = 7$ (iv) $g(0) = 11$
It is easy to see that $f(x) + g(x) = 18$ for all x. Use this fact to find $f(x)$ and $g(x)$, show your work.

Solution: Substitute $g(x)$ by $18 - f(x)$, property (i) yields $f'(x) - 2f(x) = -18$, which is linear and of 1st order. An integrating factor is $e^{\int -2\,dx} = e^{-2x}$. Multiplying both sides by e^{-2x} yields that $(e^{-2x}f(x))' = -18e^{-2x}$, and hence $e^{-2x}f(x) = 9e^{-2x} + C$. Therefore $f(x) = 9 + Ce^{2x}$. By property (iii), we have $9 + C = f(0) = 7 \Rightarrow C = -2$. Conclude that $f(x) = 9 - 2e^{2x}$ and $g(x) = 9 + 2e^{2x}$.

7. Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Find $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$

Solution:
$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$$
$$f_y(x,y) = \frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2}$$

(b) Find $f_x(0,0)$ and $f_y(0,0)$ using definition.

Solution:
$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3\cdot 0 - h\cdot 0^3}{h^2+0^2} - 0}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^3\cdot h - 0\cdot h^3}{0^2+h^2} - 0}{h} = 0$$

(c) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$

Solution:
$$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(3\cdot 0^2h - h^3)(0^2+h^2) - (0^3h - 0\cdot h^3)(2\cdot 0)}{(0^2+h^2)^2} - 0}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$
$$f_{yx}(0,0) = (f_y)_x(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(h^3 - 3h0^2)(h^2+0^2) - (h^30 - h0^3)(2\cdot 0)}{(h^2+0^2)^2} - 0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

8. Find the linear approximation of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at (3,2,6) and use it to estimate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$

Solution: The partial derivatives are
$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}},$$
and therefore the linear approximation of f at (3,2,6) is
$$L_f(3,2,6) = f(3,2,6) + f_x(3,2,6)\triangle x + f_y(3,2,6)\triangle y + f_z(3,2,6)\triangle z = 7 + \frac{3\triangle x + 2\triangle y + 6\triangle z}{7}.$$
Now, since $\triangle x = 0.02$, $\triangle y = -0.03$, $\triangle z = -0.01$, we have
$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \approx 7 + \frac{3\times 0.02 + 2(-0.03) + 6(-0.01)}{7} = 7 - \frac{0.06}{7} \approx 6.99\,.$$