
97e:33007 33B15 (26A51)
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On characterizations of the gamma function.
Math. Mag. **68** (1995), *no.* 4, 301–305.

Two known conditions characterizing the gamma function, namely $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$ for $\Gamma(x) > 0$ on $(0, \infty)$, in fact do not characterize the gamma function. Bohr and Mollerup (1922) required the additional assumption of the convexity of $\log \Gamma(x)$, a property (C) which is sufficient to characterize the gamma function. Laugwitz and Rodewald (1987) formulated a second characterization, replacing convexity of $\log \Gamma(x)$ by a so-called property (L): that $L(x) = \log \Gamma(x+1)$ satisfy $L(n+x) = L(n) + x\log(n+1) + r_n(x)$ where $r_n(x) \to 0$ as $n \to \infty$.

The author seems to be inspired by the above findings and in the process formulates a third characterization of the gamma function, and also shows the relationship between these three characterizations. From the property (L), without using the logarithm, the author formulates a characterizing property (P): $f(n+x) = f(n)n^x t_n(x)$ where $t_n(x) \to 1$ as $n \to \infty$.

By very simple calculations, in Theorem (1) we can find how the property (L) is modified by using the definitions $\Gamma(x+n) = \Gamma(n)n^x t_n(x)$, where $t_n(x) \to 1$ as $n \to \infty$, and

$$\Gamma_n(x) = \frac{n^x n!}{x(x+1)\cdots(x+n)}$$

for $x > 0$ and defined on $(0, \infty)$.

Using a definition of a pre-gamma function as one such that $f(x+1) = xf(x)$, the author shows that the properties (C), (L) and (P) are equivalent to one another for pre-gamma functions, and succeeds in proving a characterization theorem.

The reviewer is of the opinion that this attempt by the author has certainly prepared fertile new ground.

Reviewed by *P. K. Banerji*