Algebra Homework #5(due 10/26/2012)

Name: __________________ Class: __________________ Student I.D. # __________________

1. Let B be the set of upper triangular matrices in $\mathrm{GL}(2, \mathbb{Q})$, T be the set of diagonal matrices, and U be the set of matrices in B with diagonal entries 1.

   (a) Show that B, T, U are subgroups of $\mathrm{GL}(2, \mathbb{Q})$.

   (b) Show that U is normal in B, but not normal in $\mathrm{GL}(2, \mathbb{Q})$.

   (c) Show that $B = TU$.

(d) Show that the quotient group B/U is isomorphic to T.

2. Let G be a group of order 4. Show that either G is cyclic or $G = \{e, a, b, ab\}$, where a, b and ab all have order 2. Conclude that G is abelian.

3. Find all noncyclic order 4 subgroups of S_4. Which of these are normal in S_4? Give reasons.

4. Let $G = (\mathbb{Z}/mn\mathbb{Z}, +)$, where m and n are coprime integers. Let $H = \{h \in G : \text{ order } h \text{ divides } m\}$ and $K = \{k \in G : \text{ order } k \text{ divides } n\}$.

(a) Show that the intersection of H and K is $\{0\}$.

(b) Show that $H + K = G$.

(c) Show that G/H is isomorphic to K and G/K is isomorphic to H.

(d) Show that H is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and K isomorphic to $\mathbb{Z}/n\mathbb{Z}$.