Algebra Mid-term Exam

November 12, 2012

Class : ________________ Name : ________________ Student ID # : ________________

This exam is closed-book, closed-notes. **Show all your work in order to receive partial or full credit.** The exam is worth 200 points. You have 100 minutes to finish the exam.

1. True or false with reasons, 5 points each. You may quote statements proved in class or in the textbook, or from homework assignments.

   (a) The set of all positive real numbers is a group under multiplication.

   (b) The set of all positive real numbers is a group under addition.

   (c) If $a, b \in G$, where G is a group, then $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$.

   (d) If $\sigma \in S_6$, then $\sigma^n = 1$ for some $n \geq 1$.

   (e) If α, β are cycles in S_n, then $\alpha\beta = \beta\alpha$.

   (f) If σ, τ are r-cycles in S_n, then $\sigma\tau$ is an r-cycle.

   (g) If $\sigma \in S_n$ is an r-cycle, then $\alpha\sigma\alpha^{-1}$ is an r-cycle for every $\alpha \in S_n$.

   (h) An abelian group of order 4 is cyclic.

   (i) If G is a finite group and m is a divisor of $|G|$, then G contains an element of order m.

   (j) Every subgroup of S_n has order dividing $n!$.

   (k) The intersection of two cyclic subgroups of G is a cyclic subgroup.

   (l) If X is a finite subset of G, then $< X >$ is a finite subgroup.

   (m) If X is an infinite set, then $F = \{\sigma \in S_X : \sigma \text{ moves only finitely many elements of } X\}$ is a subgroup of S_X.

(n) Every proper subgroup of S_3 is cyclic.

(o) The subgroup $\{0\}$ of $\mathbb{Z}$ is isomorphic to the subgroup $\{(1)\}$ of S_5.

(p) If p is a prime, any two groups of order p are isomorphic.

(q) The subgroup $< (1\ 2) >$ is a normal subgroup of S_3.

(r) The 3-cycles (7 6 5) and (5 26 34) are conjugate in S_{100}.

2. Define a new binary operator $*$ on the integers $\mathbb{Z}$ by $x * y = x + y + 7$.

(a) Show that $(\mathbb{Z}, *)$ is a group.

(b) Is it isomorphic to $(\mathbb{Z}, +)$?

3. Let a be an element of order n in the group G.

(a) Show that if $a^k = e$, then n divides k.

(b) Show that for any integer $k > 1$, the order of the cyclic group $< a^k >$ is $\dfrac{n}{\gcd(k,n)}$.

4. Show that if every element x in a group is its own inverse then it is abelian.

5. Let $G = (\mathbb{Z}/mn\mathbb{Z}, +)$, where m and n are coprime integers. Let $H = \{h \in G : \text{ order } h \text{ divides } m\}$ and $K = \{k \in G : \text{ order } k \text{ divides } n\}$.

(a) Show that the intersection of H and K is $\{0\}$.

(b) Show that $H + K = G$.

6. Let B be the set of upper triangular matrices in $\mathrm{GL}(2, \mathbb{Q})$, T be the set of diagonal matrices, and U be the set of matrices in B with diagonal entries 1.

(a) Show that B, T, U are subgroups of $\mathrm{GL}(2, \mathbb{Q})$.

(b) Show that U is normal in B, but not normal in $\mathrm{GL}(2, \mathbb{Q})$.

(c) Show that $B = TU$.

(d) Show that the quotient group B/U is isomorphic to T.