Third Micro-century Exam for Advanced Calculus-Solutions

1. Is the function f differentiable at 0?

   (a) $f(x) = |x|^{1/2} \sin 2x$.

   Solution: A simple calculation shows

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|^{1/2}\sin 2x - 0}{x} = \lim_{x\to 0} 2|x|^{1/2} \cdot \lim_{x\to 0} \frac{\sin 2x}{2x} = 0 \cdot 1 = 0$$

   Therefore, f is differentiable at 0.

   (b) $f(x) = \begin{cases} 1 - \cos x\,, & \text{for } x \geq 0\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$

   Solution: Since we have,

$$\lim_{x\downarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\downarrow 0} \frac{(1 - \cos x) - (1 - \cos 0)}{x} = \lim_{x\downarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x(1 + \cos x)}$$

$$= \lim_{x\downarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x\downarrow 0} \frac{\sin x}{x} \cdot \lim_{x\downarrow 0} \frac{\sin x}{1 + \cos x} = 1 \cdot \frac{0}{1+1} = 0\,,$$

$$\lim_{x\uparrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\uparrow 0} \frac{0 - 0}{x} = 0\,.$$

   Therefore, $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0$ and hence f is differentiable at 0.

2. Let f be differentiable at a. Find $\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$.

   Solution: Since $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ and $\lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$

   we have,

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2} \lim_{h\to 0} \frac{f(a+h) - f(a) + f(a) - f(a-h)}{h}$$

$$= \frac{1}{2} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \frac{1}{2} f'(a) + \frac{1}{2} f'(a) = f'(a).$$

3. Consider the rational function g defined by

$$g(x) = \frac{x(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x)(8-x)(9-x)}{(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)}.$$

Find the derivative $g'(0)$ of the function g at the origin,

(a) by the definition directly;

Solution: A simple calculation shows

$$g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{\frac{x(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x)(8-x)(9-x)}{(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)} - 0}{x}$$

$$= \lim_{x\to 0} \frac{(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x)(8-x)(9-x)}{(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)} = \frac{9!}{9!} = 1$$

(b) by a suitable differentiation rule.

Solution: Write $g(x) = xf(x)$, where $f(x) = \frac{(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x)(8-x)(9-x)}{(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)}$.

By the product rule, we have $g'(x) = f(x) + xg'(x)$, and therefore

$$g'(0) = f(0) + 0 \cdot g'(0) = f(0) = \frac{9!}{9!} = 1\,.$$

4. Mean Value Theorem:

(a) State Rolle's Theorem and the Cauchy Mean Value Theorem.

Rolle's Theorem: If f is continuous on $[a,b]$ and differentiable in (a,b) satisfying $f(a) = f(b) = 0$, then there exists some $\theta \in (a,b)$ such that $f'(\theta) = 0$.

Cauchy Mean Value Theorem: If f, g are continuous on $[a,b]$ and differentiable in (a,b), then there exists some $\theta \in (a,b)$ such that

$$(\dagger) \qquad f'(\theta)[g(b)-g(a)] = g'(\theta)[f(b)-f(a)]\,.$$

(b) Using Rolle's Theorem to prove the Cauchy Mean Value Theorem.

Proof: Define the function $G(x)$ as $G(x) = \det \begin{pmatrix} f(x) & f(a) & f(b) \\ g(x) & g(a) & g(b) \\ 1 & 1 & 1 \end{pmatrix}$ $\forall x \in [a,b]$.

Then G is continuous on $[a,b]$, differentiable in (a,b) satisfying $G(a) = G(b) = 0$,

and hence by Rolle's Theorem there exists some $\theta \in (a,b)$ such that $G'(\theta) = 0$.

But, clearly we have $G'(x) = \det \begin{pmatrix} f'(x) & f(a) & f(b) \\ g'(x) & g(a) & g(b) \\ 0 & 1 & 1 \end{pmatrix}$, which yields

$G'(x) = f'(x)[g(a) - g(b)] - g'(x)[f(a) - f(b)]$, and hence $G'(\theta) = 0$ implies (†).

5. Let f be differentiable on an open interval I containing a and let

$$L(x) = f(a) + f'(a)(x-a)$$

be the linear approximation to f at a. Assume $f''(a)$ exists.

(a) Prove that $\lim_{x \to a} \dfrac{f(x) - L(x)}{(x-a)^2} = \dfrac{f''(a)}{2}$.

Proof: Since $\lim_{x \to a}[f(x) - L(x)] = \lim_{x \to a}[(x-a)^2] = 0$, L'Hôspital Rule tells us that

$$\lim_{x \to a} \frac{f(x) - L(x)}{(x-a)^2} = \lim_{x \to a} \frac{f'(x) - L'(x)}{2(x-a)} = \frac{1}{2} \lim_{x \to a} \frac{f'(x) - f'(a)}{x-a} = \frac{1}{2} f''(a) = \frac{f''(a)}{2}.$$

(b) Prove that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + R(x),$$

where $\lim_{x \to a} \dfrac{R(x)}{(x-a)^2} = 0$.

Proof: By part (a), $\lim_{x \to a} \dfrac{f(x) - L(x)}{(x-a)^2} - \dfrac{f''(a)}{2} = 0$ and hence we have

$$(\ddagger) \qquad \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2}{(x-a)^2} = 0.$$

Let $R(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2$, then we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + R(x)\,,$$

and clearly $\lim_{x \to a} \frac{R(x)}{(x-a)^2} = 0$ by (‡).

(c) Use part (b) to show that $\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a)$.

Proof: By part (b), set $x = a + h, x = a - h$ respectively, we have

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + R(a+h)\,, \text{ and}$$

$$f(a-h) = f(a) + f'(a)(-h) + \frac{f''(a)}{2}(-h)^2 + R(a-h)\,,$$

where $\lim_{h \to 0} \frac{R(a+h)}{(x-a)^2} = 0 = \lim_{h \to 0} \frac{R(a-h)}{(x-a)^2}$. So $f(a+h) - 2f(a) + f(a-h)$ becomes

$$f(a)+f'(a)h+\frac{f''(a)}{2}h^2+R(a+h)-2f(a)+f(a)+f'(a)(-h)+\frac{f''(a)}{2}(-h)^2+R(a-h)$$

which yields $f''(a)h^2 + R(a+h) + R(a-h)$, and we have

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a) + \frac{R(a+h)}{h^2} + \frac{R(a-h)}{h^2},$$

and therefore

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a) + \lim_{h \to 0} \frac{R(a+h)}{h^2} + \lim_{h \to 0} \frac{R(a-h)}{h^2}$$

which yields $f''(a) + 0 + 0 = f''(a)$. This completes the proof.