Third Micro-century Exam for Advanced Calculus

Name : _____ Student ID # : _____ Score : _____

1. Is the function f differentiable at 0?

   (a) $f(x) = |x|^{1/2} \sin 2x$.

   (b) $f(x) = \begin{cases} 1 - \cos x\,, & \text{for } x \geq 0\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$

2. Let f be differentiable at a. Find $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$.

3. Consider the rational function g defined by

$$g(x) = \frac{x(1-x)(2-x)(3-x)(4-x)(5-x)(6-x)(7-x)(8-x)(9-x)}{(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)}.$$

Find the derivative $g'(0)$ of the function g at the origin,

(a) by the definition directly;

(b) by a suitable differentiation rule.

4. Mean Value Theorem:

   (a) State Rolle's Theorem and the Cauchy Mean Value Theorem.

   (b) Using Rolle's Theorem to prove the Cauchy Mean Value Theorem.

5. Let f be differentiable on an open interval I containing a and let

$$L(x) = f(a) + f'(a)(x - a)$$

be the linear approximation to f at a. Assume $f''(a)$ exists.

(a) Prove that $\lim_{x \to a} \dfrac{f(x) - L(x)}{(x - a)^2} = \dfrac{f''(a)}{2}$.

(b) Prove that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + R(x)\,,$$

where $\lim_{x \to a} \dfrac{R(x)}{(x - a)^2} = 0$.

(c) Use part (b) to show that $\lim_{h \to 0} \dfrac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a)$.