3rd Exam for Advanced Calculus II

Name：________________ Student ID #：________________ Score：__________

1. For any $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$

   (a) Prove that $|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + \cdots + x_n^2} \cdot \sqrt{y_1^2 + \cdots + y_n^2}$.

   (b) Prove that $(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$.

2. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be continuous. Let S and T be closed subsets in $\mathbb{R}^4$ given by

$$S \subset \{(x_1, x_2, x_3, x_4) \,|\, x_1^2 + x_2^2 \leq M_1\} \,\&\, T \subset \{(x_1, x_2, x_3, x_4) \,|\, x_3^2 + x_4^2 \leq M_2\}$$

   Show that f attains a maximum and a minimum on the set $S \cap T$.

3. A set $A \subseteq \mathbb{R}^n$ is said to be convex if for each pair of points $\mathbf{a}, \mathbf{b} \in A$, the line segment joining $\mathbf{a}$ and $\mathbf{b}$ is also contained in A. This line segment is easily parameterized by

$$\mathbf{x}(t) = (1-t)\,\mathbf{a} + t\,\mathbf{b}\,.$$

Assume A and B are convex sets in $\mathbb{R}^n$.

(a) Show that $A \cap B$ is a convex set in $\mathbb{R}^n$.

(b) Show that $A + B = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B\}$ is a convex set in $\mathbb{R}^n$.

4. Differentiability: (being differentiable = having a linear approximation)

(a) Is the function $f(x, y) = x|y|$ differentiable at the point $(0, 0)$?

(b) Is the function $f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$ differentiable at the point $(0,0)$?

(c) Is the function $f(x,y) = \begin{cases} \frac{x^2y+xy}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$ differentiable at the point $(0,0)$?

5. Consider the function $f(x,y) = \dfrac{2y^2}{x^2+3xy}$.

(a) Calculate the linear approximation of f at the point $(1,-1)$.

(b) Calculate the directional derivative of the function f in the direction of $\vec{\mathbf{u}} = \dfrac{2}{\sqrt{13}}\vec{\imath} + \dfrac{3}{\sqrt{13}}\vec{\jmath}$ at the point $(1,-1)$.

6. Consider the function $f(x,y) = \dfrac{x^2+y^2}{4+y^3}$. What is the direction of steepest descent for the function f at the point $(2,1)$?