1st Exam for Advanced Calculus II

Name： ________________ Student ID #： ________________ Score： __________

1. Find the sum of the following series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1} - 3^n}{2^{2n}}$

   (c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$

2. Determine the convergence of the following series:

(a) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} 3^{-n} & \text{for } n \text{ odd} \\ 3^{-(n-2)} & \text{for } n \text{ even} \end{cases}$

(b) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

(c) $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

(d) $\sum_{n=1}^{\infty} (\sqrt{n^2+n} - n)$

(e) $\sum_{n=1}^{\infty} \frac{\log n}{n^{1.001}}$

3. Suppose that the series $\sum_n a_n$ converges absolutely and that b_n is a bounded sequence. Show that the series $\sum_n a_n b_n$ converges absolutely.

4. Let $f_n(x) = \sqrt{x^2 + \dfrac{1}{n}}$. Show that the sequence $\{f_n\}$ converges uniformly on all of $\mathbb{R}$ to $f(x) = |x|$.

5. Weierstrass M-test

   (a) State and prove Weierstrass M-test.

(b) Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(n+x)}$ of functions converges to a continuous function on the interval $[0, \infty)$.

(c) Does the series $\sum_{n=1}^{\infty} \frac{x}{1+x^2 n^2}$ of functions converge uniformly on the interval $[0, \infty)$?